代数幾何学シンポジウム記録
2008年度 pp.24-30

A counterexample to a conjecture of complete fan

Kenta Watanabe*

Abstract

If a Griffiths domain D is a symmetric Hermitian domain, the toroidal compactification of the quotient space $\Gamma\backslash D$, associated to a projective fan and a discrete subgroup Γ of $\mathrm{Aut}(D)$, was constructed by Mumford et al. Kazuya Kato and Sampei Usui studied extensions of $\Gamma\backslash D$ for a Griffiths domain D in general, and introduced a notion of "complete fan" as a generalization of a notion of projective fan. The existence of complete fans is expected. In this paper, we give an example of D which has no complete fan.

1 Introduction

Let D be a Griffiths domain, let Γ be a "neat" discrete subgroup of $\mathrm{Aut}(D)$, and let Σ be a fan consisting of rational nilpotent cones in $\mathrm{Lie}(\mathrm{Aut}(D))$ which is "strongly compatible" with Γ. Kazuya Kato and Sampei Usui [KU] introduced the notion of "polarized logarithmic Hodge structure" and enlarged the space $\Gamma\backslash D$ to the space $\Gamma\backslash D_\Sigma$ by adding the classes modulo Γ of nilpotent orbits in the directions of cones contained in Σ as the boundary points. They proved that the space $\Gamma\backslash D_\Sigma$ is the fine moduli space of polarized logarithmic Hodge structures of type $\Phi := (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_\mathbb{Z}, \langle\ ,\ \rangle, \Gamma, \Sigma)$ ([KU] 4.2.1, Theorem B), and that $\Gamma\backslash D_\Sigma$ is a "logarithmic manifold" which is nearly a complex analytic manifold but has "slits" caused by "Griffiths transversality" condition at the boundary ([KU] 4.1.1, Theorem A).

In the classical situation, that is, D is a symmetric Hermitian domain, the toroidal projective compactification $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$ was constructed with a sufficiently big fan Σ, called a projective fan, by A. Ash, D. Mumford, M. Rapoport and Y. S. Tai [AMRT].

*Department of Mathematics, Graduate School of Science, Osaka University, Osaka 560-0043, Japan. *E-mail address:* u390547e@ecs.cmc.osaka-u.ac.jp

1

For general D, Kato and Usui introduced in [KU] a "complete fan" as a generalization of a projective fan, and they gave a conjecture of the existence of such fans ([KU] 12.6.3). As an example, they gave a concrete description of the space $\Gamma\backslash D_\Sigma$ for Hodge type $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$ and for $\Sigma = \Xi$; i.e., the fan consisting of all rational nilpotent cones whose rank are less than or equal to one in Lie(Aut(D)) in [KU] 12.2.2. In this case, the fan $\Sigma = \Xi$ is complete.

In the present work, we started to generalize the description of the above example, but in fact we encounter a counterexample to the conjecture of existence of complete fans. We show that D with $h^{2,0} = h^{1,1} = h^{0,2} = 2$ has no complete fans.

After the present work, a modified version of the conjecture about complete fan is added at the end of 12.7 in [KU].

We fix a 4-tuple $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_{\mathbb{Z}}, \langle\ ,\ \rangle)$, consisting of an integer w, Hodge number $(h^{p,q})_{p,q\in\mathbb{Z}}$, a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of rank $\Sigma_{p,q}h^{p,q}$, and a non-degenerate bilinear form $\langle\ ,\ \rangle$ on $H_{\mathbb{Q}} := \mathbb{Q}\otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ which is symmetric if w is even and skew-symmetric if w is odd. Then, let D be a classifying space of polarized Hodge structure of type Φ_0 (This is also called Griffiths domain), and let $\check{D}$ be a compactdual of D.

Let

$$G_{\mathbb{Z}} := \mathrm{Aut}(H_{\mathbb{Z}}, \langle\ ,\ \rangle),$$

and for $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let

$$\begin{aligned}
&H_R := R\otimes_{\mathbb{Z}} H_{\mathbb{Z}},\ G_R := \mathrm{Aut}(H_R, \langle\ ,\ \rangle),\\
&\mathfrak{g}_R := \mathrm{Lie}(G_R)\\
&\quad = \{N \in \mathrm{End}_R(H_R) \mid \langle Nx, y\rangle + \langle x, Ny\rangle = 0 \ for\ all\ x, y \in H_R\}.
\end{aligned}$$

2 Nilpotent orbit

In this section, we recall the definition of nilpotent orbits after [KU].

We fix $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_{\mathbb{Z}}, \langle\ ,\ \rangle)$ as above.

Definition 2.1 ([KU] 0.4.2, 1.3.1) *A subset σ of $\mathfrak{g}_{\mathbb{R}}$ is said to be a* nilpotent cone, *if the following conditions are satisfied.*

(1) $\sigma = \mathbb{R}_{\geq 0}N_1 + \cdots + \mathbb{R}_{\geq 0}N_n$ *for some* $n \geq 1$ *and for some* $N_1, \ldots, N_n \in \sigma$.

(2) *Any element of σ is nilpotent as an endomorphism of $H_{\mathbb{R}}$.*

(3) $[N, N'] = 0$ *for any* N, $N' \in \sigma$ *as endomorphisms of* $H_{\mathbb{R}}$,
where $[N, N'] := NN' - N'N$.

We recall some notion about nilpotent cones in [KU] 0.4.3, 1.3.2.

A nilpotent cone is said *rational*, if we can take $N_1, \dots, N_n \in \mathfrak{g}_{\mathbb{Q}}$ in Definition 2.1 (1).

For a nilpotent cone σ, a *face* of σ is a non-empty subset τ of σ which satisfies the following two conditions.

(1) If $x, y \in \tau$ and $a \in \mathbb{R}_{\geq 0}$, then $x + y,\ ax \in \tau$.
(2) If $x, y \in \sigma$ and $x + y \in \tau$, then $x, y \in \tau$.

Definition 2.2 ([KU] 0.4.4, 1.3.3) *A* fan in $\mathfrak{g}_{\mathbb{Q}}$ *is a non-empty set* Σ *of rational nilpotent cones in* $\mathfrak{g}_{\mathbb{R}}$ *satisfying the following three conditions:*

(1) *If* $\sigma \in \Sigma$*, any face of* σ *belongs to* Σ.
(2) *If* $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ *is a face of* σ *and of* σ'.
(3) *Any* $\sigma \in \Sigma$ *is sharp. That is,* $\sigma \cap (-\sigma) = \{0\}$.

Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$. For $R = \mathbb{R}, \mathbb{C}$, we denote by σ_R the R-linear span of $\sigma \subset \mathfrak{g}_{\mathbb{R}}$.

Definition 2.3 ([KU] 0.4.7, 1.3.7) *Let* $\sigma = \Sigma_{1 \leq j \leq r}(\mathbb{R}_{\geq 0})N_j$ *be a rational nilpotent cone. A subset* Z *of* $\check{D}$ *is said to be a* σ-nilpotent orbit *if there is* $F \in \check{D}$ *which satisfies* $Z = \exp(\sigma_{\mathbb{C}})F$ *and satisfies the following two conditions.*

(1) $N_j F^p \subset F^{p-1}$ $(1 \leq j \leq r, p \in \mathbb{Z})$.
(2) $\exp(\sum_{1 \leq j \leq r} z_j N_j)F \in D$ *if* $z_j \in \mathbb{C}$ *and* $\mathrm{Im}(z_j) \gg 0$.

The conditions (1) and (2) are called *Griffiths transversality* and *positivity*, respectively.

We say that the pair (σ, F), consisting of a rational nilpotent cone $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ and of $F \in \check{D}$, generates a nilpotent orbit if $Z = \exp(\sigma_{\mathbb{C}})F$ is a σ-nilpotent orbit.

Example 2.1 Let $w = 2$, $h^{2,0} = h^{1,1} = h^{0,2} = 2$, $h^{p,q} = 0$ otherwise, and $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module with a basis $(e_j)_{1 \leq j \leq 6}$. Let $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ be the $\mathbb{Q}$-bilinear form defined by

$$(\langle e_i, e_j \rangle)_{1 \leq i,j \leq 6} = \begin{pmatrix} -1_2 & O & O \\ O & E & O \\ O & O & E \end{pmatrix}, \text{ where } 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $H'_{\mathbb{Q}} := \bigoplus_{1 \leq j \leq 4} \mathbb{Q}e_j$. For $a \in H'_{\mathbb{Q}}$, let $N_a : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ be the nilpotent endomorphism given by

3

$$N_a(b) = -\langle a, b\rangle e_5 \ \ (b \in H'_{\mathbb{Q}}), \quad N_a(e_5) = 0, \quad N_a(e_6) = a.$$

Note that, for all $a, a' \in H_{\mathbb{Q}}$, $N_a, N_{a'} \in \mathfrak{g}_{\mathbb{Q}}$ and $[N_a, N_{a'}] = 0$. Let $F \in \check{D}$ be given by $F^2 = \mathbb{C}(ie_1 + e_2) \oplus \mathbb{C}e_6$, and $F^1 = (F^2)^{\perp}$. Let $\sigma = \mathbb{R}_{\geq 0}(-N_{e_3}) + \mathbb{R}_{\geq 0}N_{e_4}$. Then, (σ, F) generates a nilpotent orbit.

Definition 2.4 ([KU] 0.4.8, 1.3.8) *Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$. As a set, we define D_Σ by*

$$D_\Sigma := \{(\sigma, Z) \mid \sigma \in \Sigma, Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit}\}.$$

Note that we have the inclusion map

$$D \hookrightarrow D_\Sigma, \ F \mapsto (\{0\}, \{F\}).$$

Definition 2.5 ([KU] 0.4.10, 1.3.10) *Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$ and let Γ be a subgroup of $G_{\mathbb{Z}}$.*

(i) *We say Γ is* compatible with Σ *if the following condition* (1) *is satisfied.*

(1) *If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then* $\mathrm{Ad}(\gamma)(\sigma) \in \Sigma$. *Here,* $\mathrm{Ad}(\gamma)(\sigma) = \gamma\sigma\gamma^{-1}$. *Note that, if Γ is compatible with Σ, Γ acts on D_Σ by*

$$\gamma : (\sigma, Z) \mapsto (\mathrm{Ad}(\gamma)(\sigma), \gamma Z) \ (\gamma \in \Gamma).$$

(ii) *We say Γ is* strongly compatible with Σ *if it is compatible with Σ and the following condition* (2) *is also satisfied. For $\sigma \in \Sigma$, define*

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma).$$

(2) *The cone σ is generated by* $\log \Gamma(\sigma)$, *that is, any element of σ can be written as a sum of c* $\log(\gamma)$ $(c \in \mathbb{R}_{\geq 0},\ \gamma \in \Gamma(\sigma))$.

Assume that Γ is "neat" and strongly compatible with Σ. $\Gamma\backslash D_\Sigma$ is a "logarithmic manifold" which is nearly a complex analytic manifold but has "slits" (see [KU]).

3 Complete fan

In this section, we recall the definition of a space D_{val} and the definition of a complete fan after [KU].

Definition 3.1 ([KU] Definition 5.3.1) *We define*

$$\mathcal{V} := \left\{ (A, V) \;\middle|\; \begin{array}{l} A \textit{ is a } \mathbb{Q}\textit{-linear subspace of } \mathfrak{g}_{\mathbb{Q}} \textit{ consisting of} \\ \textit{mutually commutative nilpotent elements,} \\ V \textit{ is a valuative submonoid of } A^* := \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q}) \\ \textit{with } V \cap (-V) = \{0\} \end{array} \right\}.$$

Here a submonoid V of A^* is said to be a valuative *submonoid, if* $V \cup (-V) = A^*$.

For $(A, V) \in \mathcal{V}$, let $\mathcal{F}(A, V)$ be the set of all rational nilpotent cones $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ satisfying the following (1) and (2).

(1) $\sigma_{\mathbf{R}} = A_{\mathbf{R}}$.
(2) Let $(\sigma \cap A)^\vee := \{h \in A^* \mid h(\sigma \cap A) \subset \mathbf{Q}_{\geq 0}\}$. Then $(\sigma \cap A)^\vee \subset V$.

Definition 3.2 ([KU] Definition 5.3.3) (i) *We define*

$$\check{D}_{\text{val}} := \left\{ (A, V, Z) \;\middle|\; \begin{array}{l} (A, V) \in \mathcal{V}, \\ Z \textit{ is an } \exp(A_{\mathbf{C}})\textit{-orbit in } \check{D} \end{array} \right\}.$$

(ii) *We define*

$$D_{\text{val}} := \left\{ (A, V, Z) \;\middle|\; \begin{array}{l} (A, V, Z) \in \check{D}_{\text{val}}, \\ \textit{there exists } \sigma \in \mathcal{F}(A, V) \textit{ such that} \\ Z \textit{ is a } \sigma\textit{-nilpotent orbit} \end{array} \right\}.$$

Definition 3.3 *Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$. For $(A, V) \in \mathcal{V}$, we define*

$$X_{A,V,\Sigma} := \{\sigma \in \Sigma \mid \sigma \cap A_{\mathbf{R}} \in \mathcal{F}(A, V)\}.$$

It is known that, if $X_{A,V,\Sigma}$ is not empty, then there exists the smallest element σ_0 of $X_{A,V,\Sigma}$ ([KU] Lemma 5.3.4).

Definition 3.4 ([KU] Definition 5.3.5) *For a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$, we define*

$$D_{\Sigma,\text{val}} := \left\{ (A, V, Z) \;\middle|\; \begin{array}{l} (A, V, Z) \in \check{D}_{\text{val}},\ X_{A,V,\Sigma} \textit{ is not empty,} \\ \exp(\sigma_{0,\mathbf{C}})Z \textit{ is a } \sigma_0\textit{-nilpotent orbit} \end{array} \right\}.$$

Here σ_0 is just as above.

Definition 3.5 ([KU] Definition 12.6.1) *A fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ is* complete, *if* $D_{\text{val}} = D_{\Sigma,\text{val}}$.

In the case where a Griffiths domain D is a symmetric Hermitian domain, a fan Σ, used in the construction of the toroidal projective compactification $G_{\mathbf{Z}} \backslash D_{\Sigma}$ in [AMRT], is complete ([KU] 12.6.4). For general D, the existence of complete fans which are strongly compatible with $G_{\mathbf{Z}}$ was expected in [KU] conjecture 12.6.3. In the next section, we give a counterexample to that conjecture.

4 Counterexample (main result)

In this section, we state our main result. Let $w = 2$, and let $h^{p,q} = 2$ $(p+q = 2,\ p,q \geq 0)$, and $h^{p,q} = 0$ otherwise. We consider about the existence of the complete fans in this case. Let $(e_j)_{1\leq j\leq 6}$ be a free basis of $H_{\mathbb{Z}}$ and $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ be the bilinear form on $H_{\mathbb{Q}}$ given by

$$(\langle e_i, e_j\rangle)_{1\leq i,j\leq 6} = \begin{pmatrix} -1_2 & O & O \\ O & E & O \\ O & O & E \end{pmatrix}, \text{ where } 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 4.1 *In this case, there is no complete fan.*

For the proof of Theorem 4.1, we first show that the rank of any rational nilpotent cone, which appears in a nilpotent orbit, is less than or equal to two. Next, assuming the existence of a complete fan Σ on D, we derive a contradiction: Σ has two different cones of rank two which have a common point as in each of their interiors.

5 Modified version of complete fan

In this section, we introduce the definition of modified version of complete fan. Recently, the definition of complete fan was modified by Chikara Nakayama as follows.

Definition 5.1 (Chikara Nakayama) *Let N be a set of all rational nilpotent cones which appear in a nilpotent orbit. Then, a fan Σ in $\mathfrak{g}_{\mathbb{Q}}$ is said to be* complete *if it satisfies following condition.*

$$\bigcup_{\sigma\in\Sigma}\sigma = \bigcup_{\sigma\in N}\sigma$$

By this definition, the conjecture of the existence of complete fan was modified as follows.

Conjecture 5.1 *There exists a fan in $\mathfrak{g}_{\mathbb{Q}}$ which satisfies the condition in* Definition 5.1, *and is strongly compatible with $G_{\mathbb{Z}}$.*

References

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. S. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, Brookline, 1975.

[CK] E. Cattani and A. Kaplan, *Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure*, Invent. Math. 67 (1982), 101-115.

[CKS] E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures,* Ann. of Math. 123 (1986), 457-535.

[D] P. Deligne, *La conjecture de Weil, II,* Publ. Math. I.H.E.S. 52 (1980), 137-252.

[G] P. A. Griffiths, *Periods of integrals on algebraic manifolds. I. Construction and properties of modular varieties.* Amer. J. Math. 90 (1968), 568-626.

[KU] K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures,* Ann. Math. Studies, Princeton Univ. Press, Princeton, 2008 (in press).

[S] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. 22 (1973), 211-319.